Analysis of Linear Expansivity of Metallic Strips Joined Linearly and with Circular Joints

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Abstract

In this paper the linear expansivity of a metallic strip that are joined linearly and with circular joints were analyzed. The Linear expansivity equations were converted into Integro-differential equation (IDE) and the resulting equation are then solved using Variational Iteration Method (VIM). The results obtained were also analysed for various parameters (temperature, length and time) at a specific rate of change in temperature. It was observed that both the temperature and length of the metallic strip are directly proportional to time.

Keywords: Linear Expansivity, Variational Iteration Method, Metallic Strips, Circular Joint

1.0 Introduction

Metallic strip is used to convert a temperature change into mechanical displacement. The strip consists of two or more strips of different metals which expand at different rates as they are heated. The strips are joined together throughout their length by riveting, brazing or welding. The different expansions force the flat strip to bend one way if heated, and in the opposite direction if cooled below its initial temperature. The metal with the higher coefficient of thermal expansion is on the outer side of the curve when the strip is heated and on the inner side when cooled. Thermal expansion joints in the Auckland Harbour Bridge in New Zealand allow bridges to change length without buckling Ingolfson Wikimedia Commons (2012). Many works have been carried out by various researchers they include: Wang et. al. (2006), Eric and Eugeniu (2002), Jean-Loïc et. al. (2002), Lienhard et.al. (2008). This paper focused on examining the effect of temperature on the linear expansivity of stripped metals (Iron, Copper and Brass) with the view of analyzing the effect of the heat transfer overt time (t) at a specific rate of increase in temperature.

2.0 Mathematical Formulation of Problem

The formula for linear expansivity is given by

$$\Delta l = \alpha\, l\, \Delta T \tag{1}$$

were, Δl is the change in length, ΔT is the change in Temperature of Iron, α is the linear expansivity coefficient and l is the length.

$$\frac{\Delta l}{\Delta T} = \alpha l \tag{2}$$

$$\lim_{t\to 0}\left(\frac{\Delta l}{\Delta t}\right) = \alpha l \left(\lim_{t\to 0}\left(\frac{\Delta T}{\Delta t}\right)\right)$$

we obtain

$$\frac{dl}{dt} = \alpha l\left(\frac{dT}{dt}\right) \tag{3}$$

But

$$\frac{dT}{dt} = KT \Rightarrow T = \frac{1}{K}\left(\frac{dT}{dt}\right) \tag{4}$$

hence Integrating (3) by part and considering (4) we have,

$$\frac{dl}{dt} = K\left[l - l_0 + \alpha^2 \int_0^t lT\left(\frac{dT}{dt}\right)dt\right] \tag{5}$$

Where K is the rate of increase in temperature due to time and l_0 is the initial length.

Problem 1. Heat Transfer of Metallic Strips (Iron, Copper and Brass) that are joined linearly

$$\Delta T^i = \alpha \Delta L \tag{6}$$

Where ΔT^i is the change in temperature for iron and α is the linear expansivity coefficient of iron.

$$\Delta T^c = \beta \Delta L \tag{7}$$

Where ΔT^c is the change in temperature for copper and β is the linear expansivity coefficient of copper.

$$\Delta T^b = \gamma \Delta L \tag{8}$$

Where ΔT^b in the change in temperature for brass and γ is the linear expansivity coefficient of brass.

Thereby it results into the following differential equation for temperatures

$$\frac{dT^i}{dt} = K_1 T^i + K_2 T^c \tag{9}$$

$$\frac{dT^c}{dt} = K_2 T^c + K_3 T^b \tag{10}$$

$$\frac{dT^b}{dt} - K_3 T^b = 0 \tag{11}$$

and for the length we have:

The linear expansivity from Brass

$$\frac{dB}{dt} = K_3\left[B - B_0 + \gamma^2 \int_0^t BT^b\left(\frac{dT^b}{dt}\right)\right]dt \tag{12}$$

The linear expansivity from Copper to Brass

$$\frac{dC}{dt} = K_2\left[C - C_0 + \beta^2 \int_0^t CT^c\left(\frac{dT^c}{dt}\right)dt\right] + K_3\gamma^2 \int_0^t BT^b\left(\frac{dT^b}{dt}\right)dt \tag{13}$$

The linear expansivity from Iron to Copper

$$\frac{dI}{dt} = K_1\left[I - I_0 + \alpha^2 \int_0^t IT^i\left(\frac{dT^i}{dt}\right)dt\right] + K_2\beta^2 \int_0^t CT^c\left(\frac{dT^c}{dt}\right)dt \tag{14}$$

Solutions to Eqn. (9) - (14) are obtained using Variational Iteration Method (He, 1999-2006 and He et. al 2007):
Using VIM to solve for Eqn. (11), we have;

$$T_{n+1}^{b} = T_0^b e^{K_3 t} - \int_0^t e^{K_3(s-t)} \cdot 0 ds \tag{15}$$

$T_0^b(t) = T_0^b e^{K_3 t}$ Initial approximation for temperature of Brass

With the solution

$$T^b(t) = T_0^b e^{K_3 t} \tag{16}$$

Using VIM to solve for Eqn. (10), we have;

$$T_{n+1}^{c} = T_0^c e^{K_2 t} + K_3 \int_0^t e^{K_2(s-t)} T_0^c e^{K_3 s} ds \tag{17}$$

$T_0^c(t) = T_0^c e^{K_2 t}$ Initial approximation for temperature of Copper

With the solution

$$T^c(t) = T_0^c e^{K_2 t} + \frac{K_3 T_0^b}{K_2 + K_3}\left(e^{K_3 t} - e^{-K_2 t}\right) \tag{18}$$

Using VIM to solve for Eqn. (9), we have;

$$T_{n+1}^{i} = T_0^i e^{K_1 t} + K_2 \int_0^t e^{K_1(s-t)} T_0^c e^{K_2 s} ds \tag{19}$$

$T_0^i(t) = T_0^i e^{K_1 t}$ Initial approximation for temperature of Iron

With the solution

$$T^i(t) = T_0^i e^{K_1 t} + \frac{K_2 T_0^c}{K_1 + K_2}\left(e^{K_1 t} - e^{-K_2 t}\right) \tag{20}$$

Solving Eqn. (12) using VIM we have

$$B_{n+1}(t) = B_0 e^{K_3 t} - K_3 \int_0^t \left[e^{K_3(s-t)} \left(B_0 - \gamma^2 \int_0^{\xi} B_0 T^b \left(\frac{dT^b}{dt} \right) \right) ds \right] d\xi \tag{21}$$

$B_0(t) = B_0 e^{K_3 t}$ Initial approximation for the length of Brass

With the solution

$$B(t) = \frac{1}{12} B_0 \left[12 e^{K_3 t} - 12 + 12 e^{-K_3 t} - \gamma^2 \left(T_0^b\right)^2 e^{3K_3 t} + 3\gamma^2 \left(T_0^b\right)^2 e^{-K_3 t} - 4\gamma^2 \left(T_0^b\right)^2 \right] \tag{22}$$

Solving Eqn. (13) using VIM, we have;

$$C_{n+1}(t) = C_0 e^{K_2 t} - K_2 \int_0^t e^{K_2(s-t)} \left[\left(C_0 - \beta^2 \int_0^{\xi} C_0 T_0^c \left(\frac{dT_0^c}{dt} \right) ds \right) - K_3 \gamma^2 \int_0^{\xi} B_0 T_0^b \left(\frac{dT_0^b}{dt} \right) ds \right] d\xi \tag{23}$$

$C_0(t) = C_0 e^{K_2 t}$ Initial approximation for the length of Copper

With the solution

$$C(t)=\left(\frac{C_0}{12(K_2+3K_3)}\right)\Big(12K_2e^{K_2t}-12K_2+12K_2e^{-K_2t}+36K_3e^{K_2t}-36K_3+36K_3e^{-K_2t}-4K_2\beta^2\left(T_0^c\right)^2$$
$$-12K_3\beta^2\left(T_0^c\right)^2+3K_2\beta^2\left(T_0^c\right)^2e^{-K_2t}+9K_3\beta^2\left(T_0^c\right)^2e^{-K_2t}+K_2\beta^2\left(T_0^c\right)^2e^{3K_3t}+3K_3\beta^2\left(T_0^c\right)^2e^{3K_2t}\Big)$$
$$+B_0\left(12K_3^2\gamma^2\left(T_0^b\right)^2e^{-K_2t}-4K_2K_3\gamma^2\left(T_0^b\right)^2-12K_3^2\gamma^2\left(T_0^b\right)^2+4K_2K_3\gamma^2\left(T_0^b\right)^2e^{3K_3t}\right) \tag{24}$$

Solving Eqn. (14) we have;

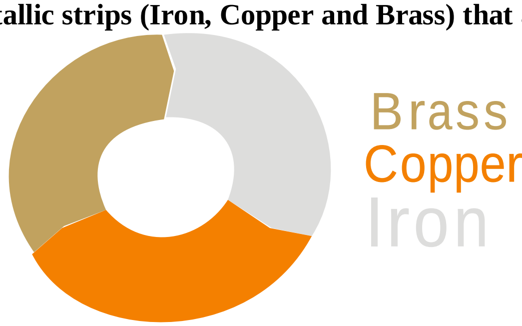
$$I_{n+1}(t)=I_0e^{K_1t}-K_1\int_0^t e^{K_1(s-t)}\left[\left(I_0-\alpha^2\int_0^{\xi}I_0T_0^i\left(\frac{dT_0^i}{dt}\right)ds\right)-K_2\beta^2\int_0^{\xi}C_0T_0^c\left(\frac{dT_0^c}{dt}\right)ds\right]d\xi \tag{25}$$

$I_0(t)=I_0e^{K_1t}$ Initial approximation for the length of Iron

With the solution

$$I(t)=\left(\frac{I_0}{12(K_1+3K_2)}\right)\Big(12K_1e^{K_1t}-12K_1+12K_1e^{-K_1t}+36K_2e^{K_1t}-36K_2+36K_2e^{-K_1t}-4K_1\alpha^2\left(T_0^i\right)^2$$
$$-12K_2\alpha^2\left(T_0^i\right)^2+3K_1\alpha^2\left(T_0^i\right)^2e^{-K_1t}+9K_2\alpha^2\left(T_0^i\right)^2e^{-K_1t}+K_1\alpha^2\left(T_0^i\right)^2e^{3K_2t}+3K_2\alpha^2\left(T_0^i\right)^2e^{3K_1t}\Big)$$
$$+C_0\left(12K_2^2\beta^2\left(T_0^c\right)^2e^{-K_1t}-4K_1K_2\beta^2\left(T_0^c\right)^2-12K_2^2\beta^2\left(T_0^c\right)^2+4K_1K_2\beta^2\left(T_0^c\right)^2e^{3K_2t}\right) \tag{26}$$

Problem 2. Heat transfer of metallic strips (Iron, Copper and Brass) that are joined in circular form.



Similarly the temperature equations are

$$\frac{dT^i}{dt}=K_3T^i+K_1T^b \tag{27}$$

$$\frac{dT^c}{dt}=K_2T^c+K_3T^i \tag{28}$$

$$\frac{dT^b}{dt}=K_1T^b+K_2T^c \tag{29}$$

and length equations are:

The linear expansivity from Brass to Copper

$$\frac{dB}{dt}=K_1\left[B-B_0+\alpha^2\int_0^t BT^b\left(\frac{dT^b}{dt}\right)dt\right]+K_2\beta^2\int_0^t CT^c\left(\frac{dT^c}{dt}\right)dt \tag{30}$$

The linear expansivity from Copper to Iron

$$\frac{dC}{dt} = K_2\left[C - C_0 + \beta^2\int_0^t CT^c\left(\frac{dT^c}{dt}\right)dt + K_3\gamma^2\int_0^t IT^i\left(\frac{dT^i}{dt}\right)dt\right] \tag{31}$$

The linear expansivity from Iron to Brass

$$\frac{dI}{dt} = K_3\left[I - I_0 + \gamma^2\int_0^t IT^i\left(\frac{dT^i}{dt}\right)dt\right] + K_1\alpha^2\int_0^t BT^b\left(\frac{dT^b}{dt}\right)dt \tag{32}$$

also Using the VIM He. et. al. (2007) to solve Eqn. (27) – (28) we obtain

$$T^b(t) = T_0^b e^{K_1 t} + \frac{K_2 T_0^c}{K_1 + K_2}\left(e^{K_2 t} - e^{-K_1 t}\right) \tag{33}$$

$$T^c(t) = T_0^c e^{K_2 t} + \frac{K_3 T_0^i}{K_2 + K_3}\left(e^{K_3 t} - e^{-K_2 t}\right) \tag{34}$$

$$T^i(t) = T_0^i e^{K_3 t} + \frac{K_1 T_0^b}{K_3 + K_1}\left(e^{K_1 t} - e^{-K_3 t}\right) \tag{35}$$

$$\begin{aligned} B(t) = \frac{B_0}{12(K_1 + 3K_2)}&\Big(12K_1e^{K_1t} - 12K_1 + 12K_1e^{-K_1t} + 36K_1e^{K_1t} - 36K_1 + 36K_1e^{-K_1t} + K_1\alpha^2\left(T_0^b\right)^2e^{3K_1t} \\ &+ 3K_2\alpha^2\left(T_0^b\right)^2e^{3K_1t} + 3K_1\alpha^2\left(T_0^b\right)^2e^{-K_1t} + 9K_2\alpha^2\left(T_0^b\right)^2e^{-K_1t} - 4K_1\alpha^2\left(T_0^b\right)^2 - 12K_2\alpha^2\left(T_0^b\right)^2\Big) \\ &+ C_0\Big(4K_1K_2\beta^2\left(T_0^c\right)^2e^{3K_2t} - 4K_1K_2\beta^2\left(T_0^c\right)^2 - 12K_2^2\beta^2\left(T_0^c\right)^2 + 12K_2^2\beta^2\left(T_0^c\right)^2e^{-K_1t}\Big) \end{aligned} \tag{36}$$

$$\begin{aligned} C(t) = \frac{C_0}{12(K_2 + 3K_3)}&\Big(12K_2e^{K_2t} - 12K_2 + 12K_2e^{-K_2t} + 36K_2e^{K_2t} - 36K_2 + 36K_2e^{-K_2t} + K_2\beta^2\left(T_0^c\right)^2e^{3K_2t} \\ &+ 3K_3\beta^2\left(T_0^c\right)^2e^{3K_2t} + 3K_2\beta^2\left(T_0^c\right)^2e^{-K_2t} + 9K_3\beta^2\left(T_0^c\right)^2e^{-K_2t} - 4K_2\beta^2\left(T_0^c\right)^2 - 12K_3\beta^2\left(T_0^c\right)^2\Big) \\ &+ I_0\Big(4K_2K_3\gamma^2\left(T_0^i\right)^2e^{3K_3t} - 4K_2K_3\gamma^2\left(T_0^i\right)^2 - 12K_3^2\gamma^2\left(T_0^i\right)^2 + 12K_3^2\gamma^2\left(T_0^i\right)^2e^{-K_2t}\Big) \end{aligned} \tag{37}$$

$$\begin{aligned} I(t) = \frac{I_0}{12(K_3 + 3K_1)}&\Big(12K_3e^{K_3t} - 12K_3 + 12K_3e^{-K_3t} + 36K_3e^{K_3t} - 36K_3 + 36K_3e^{-K_3t} + K_3\gamma^2\left(T_0^i\right)^2e^{3K_3t} \\ &+ 3K_1\gamma^2\left(T_0^i\right)^2e^{3K_3t} + 3K_3\gamma^2\left(T_0^i\right)^2e^{-K_3t} + 9K_1\gamma^2\left(T_0^i\right)^2e^{-K_3t} - 4K_3\gamma^2\left(T_0^i\right)^2 - 12K_1\gamma^2\left(T_0^i\right)^2\Big) \\ &+ B_0\Big(4K_1K_3\alpha^2\left(T_0^b\right)^2e^{3K_1t} - 4K_1K_3\alpha^2\left(T_0^b\right)^2 - 12K_1^2\alpha^2\left(T_0^b\right)^2 + 12K_1^2\alpha^2\left(T_0^b\right)^2e^{-K_3t}\Big) \end{aligned} \tag{38}$$

Equation (33) - (35) are final temperatures for Brass, Copper and Iron respectively were Eqn. (36) - (38) are final Length of Brass, Copper and Iron respectively.

3. Numerical Applications

We present graphical summaries by considerring the following properties for the linear expansivity of a metallic strip,

$\alpha = 12.0\times10^{-6}m/mK$, $\beta = 1.87\times10^{-6}m/mK$, $\gamma = 1.66\times10^{-6}m/mK$

$T^i(0) = 273$, $T^c(0) = 273$, $T^b(0) = 273$
$B(t) = 0.1$, $C(t) = 0.1$, $B(t) = 0.1$

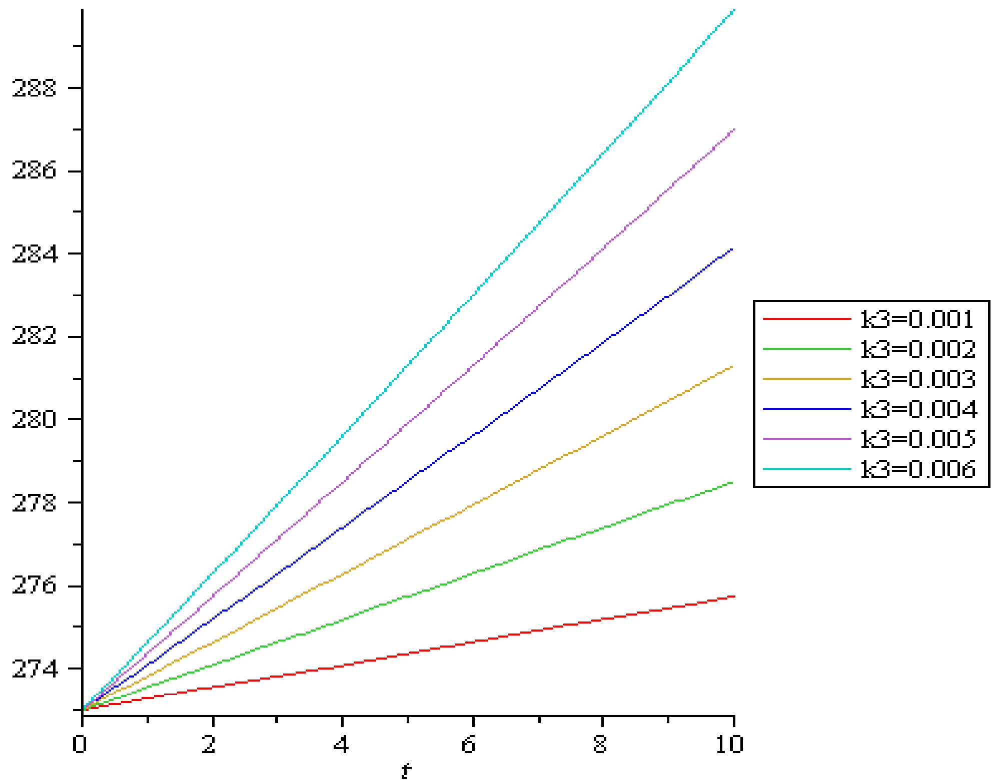


Figure 4.1 Graph of Temperature of Brass against time

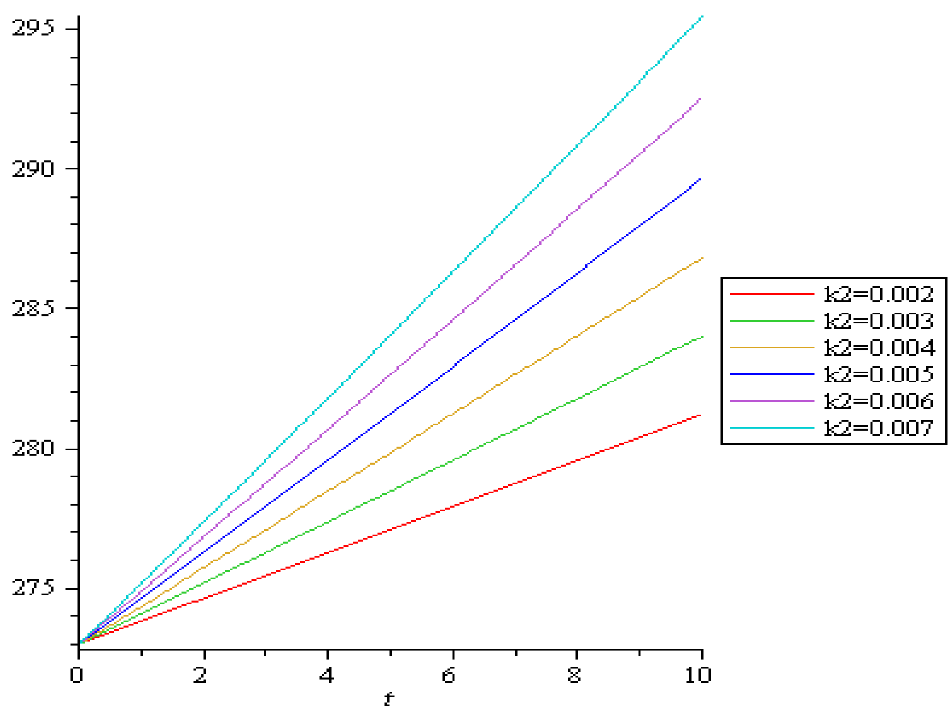


Figure 4.2 Graph of Temperature Copper against time

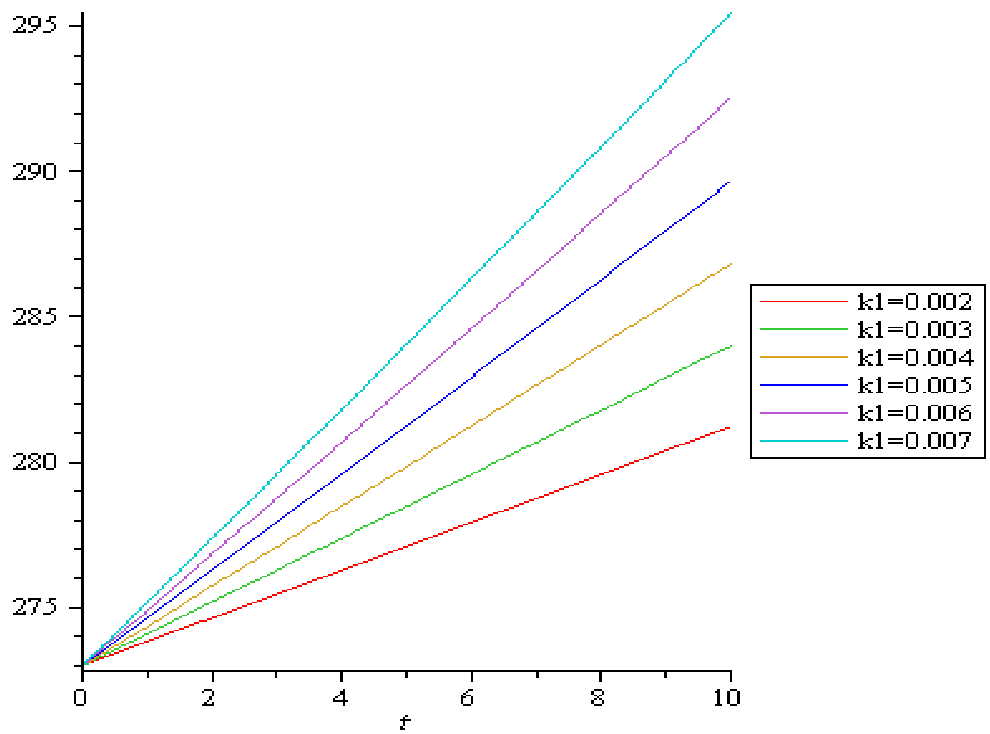


Figure 4.3 Graph of Temperature of Iron against time

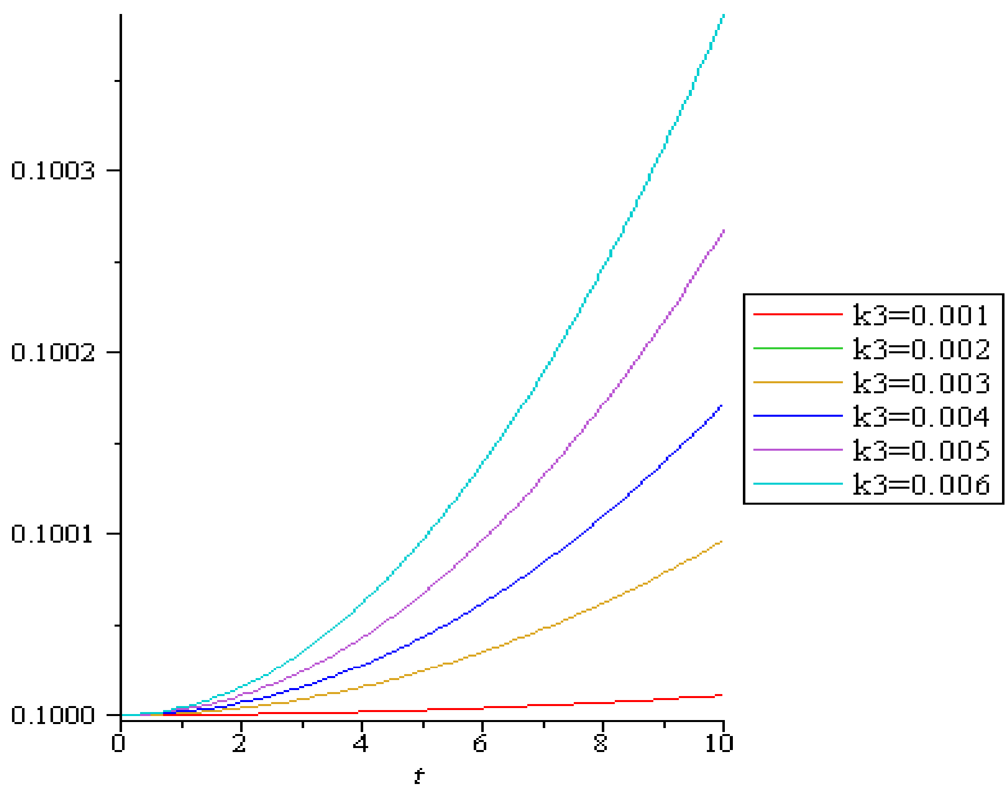


Figure 4.4 Graph of Length of Brass against time

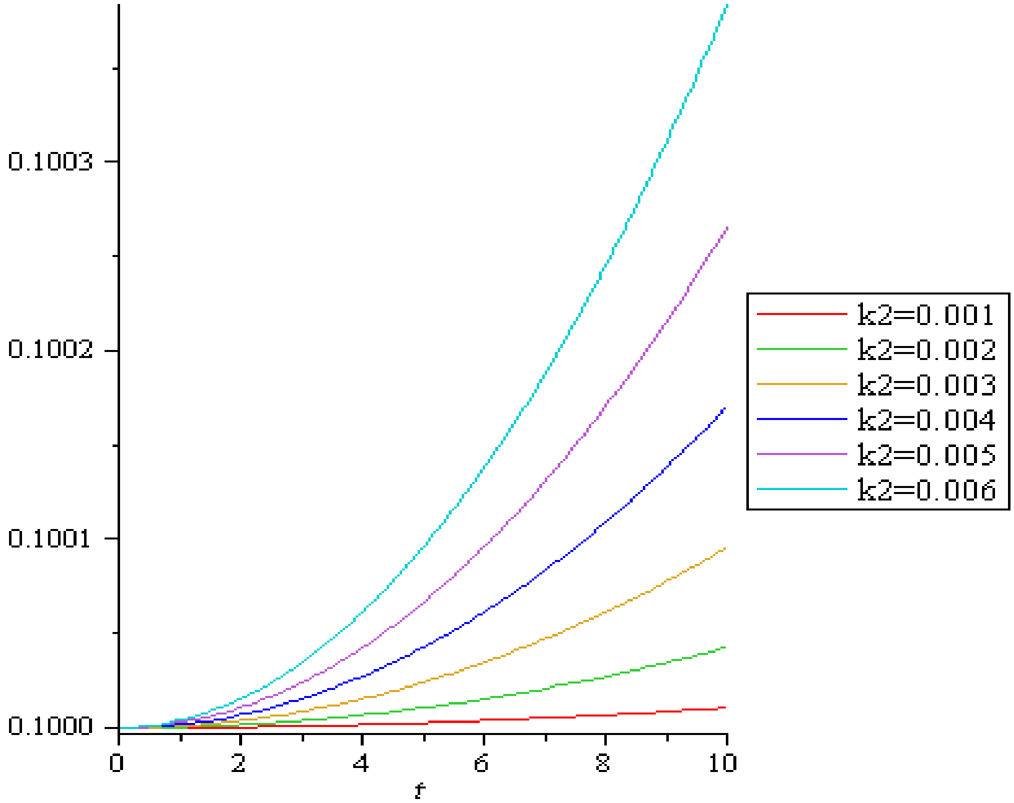


Figure 4.5 Graph of length of Copper against time.

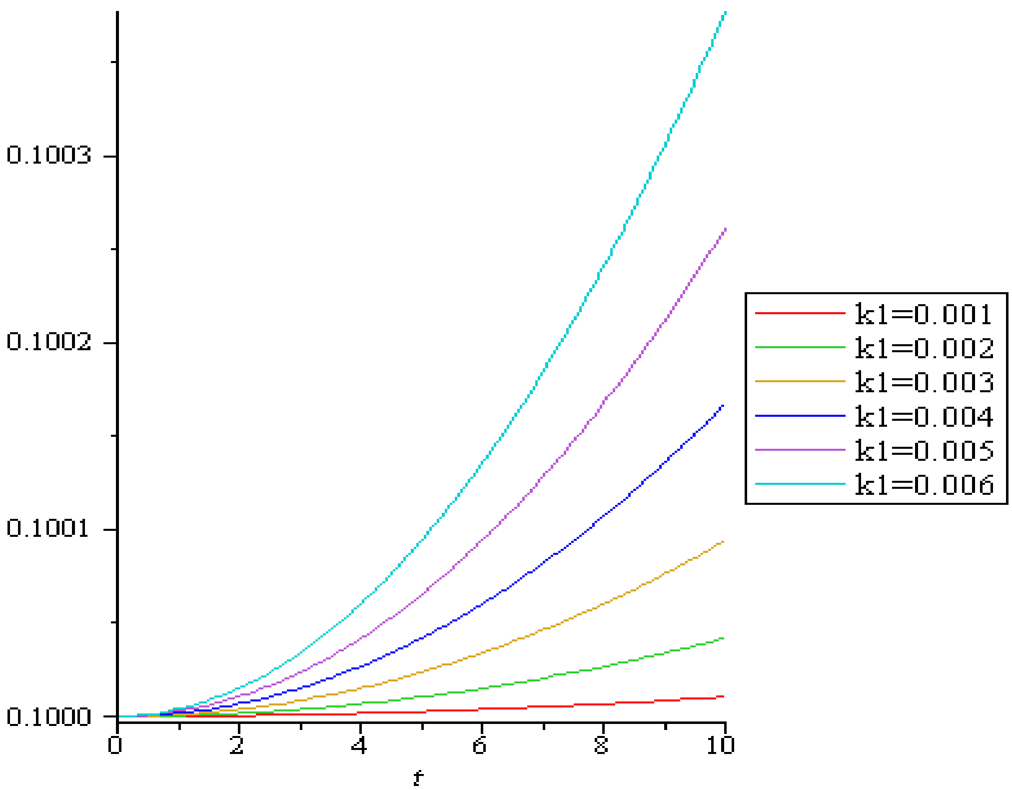


Figure 4.6 Graph of Length of Iron against time.

4.0 Discussion of Results

Three stripped metals (Iron, Copper and Brass) are considered it can be observed that iron has the highest coefficient of thermal expansion, followed by copper and brass. This implies that when heated, iron will be at the outer side of the curve, copper at the middle and brass on the inner side. From figure 4.1 to figure 4.3 it is observed that the temperature is directly proportional to the time. Also from figure 4.4 to figure 4.6 it is deduced that as time increases the length of the metals also increases thereby making the heat transfer from one metal to the other faster.

5.0 Summary

In conclusion, the use of variational iteration method has been a very useful technique for solving integro-differential equation which yield solutions that converges to that of exact solutions. The study analyzes the temperature and length of metals while varying the temperature coefficient and time. To this effect it is noted that the increase in length of the metals and the transfer of heat from one metal to the other is directly proportional to the increase in the temperature over time.

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