Total Coloring and Total Coloring of Thorny Graphs

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Abstract

Over the 150 years various works have been done on the coloring of graphs such as vertex coloring, edge coloring. The last 40 years total coloring of the graphs has been considered by various authors. In this paper we give the total chromatic number of the thorny graphs and some theorems about total chromatic number of thorny graphs.

Key Words: Total coloring, total chromatic number, thorny graphs.

1. Introduction

A total coloring of a graph G is a coloring of all elements of G, i.e. vertices and edges, so that no two adjacent or indecent elements receive the same color. The minimum number of colors is called the total chromatic number $\chi_T(G)$ of G. [Isobe, Zhou & Nishizeki, 1998]

Total coloring was introduced by Behzad and Vizing in numerous occasions between 1964 and 1968. [Jensen & Toft, 1995] Behzad stated the total chromatic number of several classes of graphs in his Ph.D. thesis called "Graphs and their chromatic numbers".

For any graph G^* that can be obtained from a parent connected graph G by attaching $p_i \geq 0$ new vertices of degree one to each vertex i, is called thorny graph. [Gutman, 1998] In this paper we give the total chromatic number of the thorny graphs where $p_i = r$ ($r \in Z^+$).

The thorny graph G^* of G is used to design communication networks and to represent molecular structure. [Dundar, 2003, Dundar, Aytaç & Dundar, 2002] In this study we give some theorems about total chromatic number of thorny graphs.

- We consider (simple) graphs which are finite, undirected with no loops or multiple edges.
- Let G be such a graph, $V = V(G)$ its vertex set, $E = E(G)$ its edge set, and let its vertices, whose number is n, be labeled $v_1, v_2, ..., v_n$.
- We follow the notations of Chartrand and Lesniak (2005).

Theorem 1.1 [Jensen & Toft, 1995] The Total Coloring Conjecture (Behzad, Vizing) is,
$\chi_T(G) \leq \Delta(G) + 2$

Theorem 1.2 [Villa, 2005] The total chromatic number of P_n is,
$\chi_T(P_n) = \Delta(P_n) + 1 = 3$

Theorem 1.3 [Colbourn & Mahmoodian, 1995] The total chromatic number of C_n is,

$$\chi_T(C_n) = \begin{cases} 3, & n = 0 \ (\text{mod}\, n = 3) \\ 4, & otherwise \end{cases}$$

Theorem 1.4 [Saetung & Chumni, 2010] The total chromatic number of K_n is,

$$\chi_T(K_n) = \begin{cases} \Delta(K_n) + 2, & n = 2k, \ n = 1,\dots,n \\ \Delta(K_n) + 1, & n = 2k+1, \ n = 1,\dots,n \end{cases}$$

2. *Total Coloring Of Thorny Graphs*

In this chapter, we give some of the theorems about total chromatic number of thorny graphs.

Theorem 2.1 The total chromatic number of thorny graph G^* of G is,
$\chi_T(G^*) \geq \Delta(G^*) + 1$.

Proof: For total coloring of the thorny graph G^* with the minimum color, the maximum degree vertex and the indecent edge(s) to this vertex must be colored. The maximum degree of thorny graph G^* is $\Delta(G) + r$. For edge coloring $\Delta(G) + r$ colors are enough and one more color is needed to color a vertex indecent to these edges. If the graph is tree, total chromatic number of thorny graph of tree is $\Delta(T^*) + 1$. So the total chromatic number of the thorny graph G^* is at least one more than the maximum degree of G^*.

Corollary 2.1.1 The total chromatic number of thorny path P_n^* is,
$\chi_T(P_n^*) = r + 3$. $(n \geq 3)$

Proof: For total coloring of the thorny graph of P_n^* with the minimum color, the maximum degree vertex and the indecent edge(s) to this vertex must be colored. The maximum degree of graph P_n^* is $(r+2)$ for $(n \geq 3)$. For edge coloring $(r+2)$ colors are enough and one more color is needed to color a vertex indecent to these edges. So $(r + 2 + 1 = r + 3)$, $\chi_T(P_n^*) = r + 3$. And the maximum degree of graph P_2^* is $(r+1)$. So $\chi_T(P_2^*) = r + 2$.

Corollary 2.1.2 The total chromatic number of thorny cycle C_n^* is,
$\chi_T(C_n^*) = r + 3$.

Proof: For total coloring of the thorny cycle C_n^* with the minimum color, the maximum degree vertex and the indecent edge(s) to this vertex must be colored. The maximum degree of graph C_n^* is $(r+2)$. For edge coloring $(r+2)$ colors are enough and one more color is needed to color a vertex indecent to these edges. So $(r + 2 + 1 = r + 3)$, $\chi_T(C_n^*) = r + 3$.

Corollary 2.1.3 The total chromatic number of thorny complete graph K_n^* is,
$\chi_T(K_n^*) = n + r$.

Proof: For total coloring of the thorny complete graph K_n^* with the minimum color, the maximum degree vertex and the indecent edge(s) to this vertex must be colored. The maximum degree of graph K_n^* is $(n+r-1)$. For edge coloring $(n+r-1)$ colors are enough and one more color is needed to color a vertex indecent to these edges. So $(n+r-1+1=n+r)$, $\chi_T(K_n^*)=n+r$.

Theorem 2.2 For every graph G,

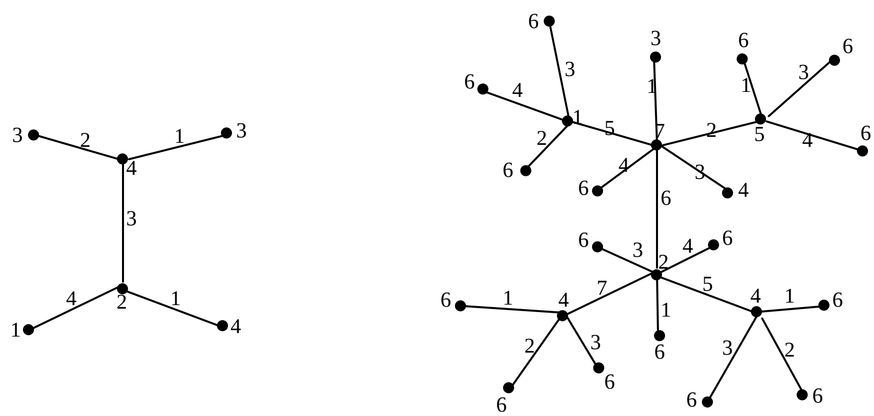
$$\chi_T(G) \le \chi_T(G^*) \le \chi_T(G)+r$$

Proof: Since the inequality $\chi_T(G) \le \chi_T(G^*)$ is clear, we show that $\chi_T(G^*) \le \chi_T(G)+r$. Suppose that $\chi_T(G)=k$. Construct G^* by adding new r-vertices $(u_1,u_2,u_3,\dots,u_r)$ to G and joining these vertices to every vertex in G. Since the $(k+r)$-total coloring number of G^* is defined. So the total chromatic number of G^* is at most $(k+r)$.

Corollary 2.2.1 i) $\chi_T(P_n^*) = \chi_T(P_n)+r$

ii) $\chi_T(C_n^*) \le \chi_T(C_n)+r$

iii) $\chi_T(K_n^*) \le \chi_T(K_n)+r$



$\chi_T(T)=4$ $\chi_T(T^*)=7$

Figure 2.1 Total choromatic number of Tree T and Thorny Tree T^* $(r=3)$.

Theorem 2.3 Let G^* be a thorny graph of G. If v is a vertex of G^*, then

$$\chi_T(G^*)-2 \le \chi_T(G^*-v)$$

Proof: We verify the lower bound for $\chi_T(G^*-v)$. Suppose that $\chi_T(G^*-v)=k$. Then the total coloring of G^* defined by

$$\chi_T(G^*) = \begin{cases} k+i, \text{ if } \deg v = \Delta(G^*) \ (1 \le i \le 2) \\ k, \textit{otherwise} \end{cases}$$

a total coloring of G^* is at most $k+2$. Therefore, $\chi_T(G^*) \le k+2 = \chi_T(G^*-v)+2.$

Theorem 2.4 If G is a graph of order n, then

(i) $2\sqrt{r+n} \le \chi_T(G^*) + \chi_T(\overline{G}^*) \le n+2r+4$

(ii) $r+n \le \chi_T(G^*).\chi_T(\overline{G}^*) \le \left(\frac{n}{2}+r+2\right)^2$

Proof: First we verify the lower bounds for $\chi_T(G^*) + \chi_T(\overline{G}^*)$ and $\chi_T(G^*).\chi_T(\overline{G}^*)$. To color G and $\overline{G}$, we obtain a coloring of K_n. The total chromatic number of the thorny graph $K_n{}^*$ of K_n is ($n+r$). Thus, this is a total chromatic number of $K_n{}^*$ using at most $\chi_T(G^*).\chi_T(\overline{G}^*)$ colors, so

$$r+n = \chi_T(K_n{}^*) \le \chi_T(G^*).\chi_T(\overline{G}^*)$$

This establishes the lower bound in (ii). Since the arithmetic mean of two positive numbers is always at least as large as their geometric mean, we have

$$\sqrt{r+n} \le \sqrt{\chi_T(G^*).\chi_T(\overline{G}^*)} \le \frac{\chi_T(G^*) + \chi_T(\overline{G}^*)}{2}$$

This verifies the lower bound of (i).

To verify the upper bound (i), according to the theorem "The Total Coloring Conjecture" $\chi_T(G) \le \Delta(G)+2$. Thorny graph G^* has a maximum vertex degree (n-1+r). So $\overline{G}^*$ has maximum vertex degree (1+r).

$$\chi_T(G^*) + \chi_T(\overline{G}^*) \le (n-1+r+2)+(1+r+2) = n+4+2r$$

This is the proof of the upper bound in (i). And

$$\sqrt{\chi_T(G^*).\chi_T(\overline{G}^*)} \le \frac{\chi_T(G^*) + \chi_T(\overline{G}^*)}{2} \le \frac{n+4+2r}{2}$$

$$\chi_T(G^*).\chi_T(\overline{G}^*) \le \left(\frac{n}{2}+r+2\right)^2$$

this is the proof of the upper bound in (ii).

Theorem 2.5 For every thorny graph G^* of G is,

$$\log_2\left[\chi_T(G^*)+1\right] \le \chi_T(G),\ (r \le \chi_T(G)+1)$$

Proof: Let $\chi_T(G)=k$ and let there be given a set k-coloring of G using the colors in N_k. Since there are 2^k-1 nonempty subsets of N_k, it follows that G^* uses at most 2^k-1 colors.

$$X_T(G^*) \le 2^k-1 \quad \Longrightarrow \quad X_T(G^*)+1 \le 2^k \quad \Longrightarrow \quad \log_2\left[\chi_T(G^*)+1\right] \le \log_2 2^k = k = \chi_T(G)$$

$$\log_2\left[\chi_T(G^*)+1\right] \le \chi_T(G)\ (r \le \chi_T(G)+1).$$

3. Algorithm for Total Chromatic Number of a Thorny Graph

In this chapter, we develop an algorithm to find a total coloring of a given Thorny Graph G^*.

A0: Start with a graph G^* and a list of colors 1, 2, 3, …
A1: Identify the vertex that has maximum degree,
A2: Color this vertex and edges incident to this vertex with different colors from the list,
A3: Color the uncolored vertices and incident edges to these vertices with the used colors. If necessary chose a different color from the list,
A4: Repeat A3 until all vertices and edges are colored,
A5: Stop.

Conclusions

In this study, total coloring and total chromatic number for thorny graphs of some of the most common classes of graphs are discussed. It is not easy to evaluate the total chromatic number of graphs. The most common classes of graphs' total chromatic number are used to evaluate the total chromatic number of graphs. Through this idea we used some of the total chromatic number of certain graphs to evaluate the total chromatic number of thorny graphs.

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